PHILOSOPHICAL TRANSACTIONS.

I. *Note on Professor* SYLVESTER'S *representation of the Motion of a free rigid Body by that of a material Ellipsoid whose centre is fixed, and which rolls on a rough Plane.*

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IN a paper published in the Transactions of the Royal Society for 1866, Professor SYLVESTER has given an important extension of POINSOT'S representation of the motion of a freely rotating rigid body, by means of the momental ellipsoid. He has proved that if a material ellipsoid, similar in form to the momental ellipsoid, and so constituted that its principal moments of inertia, A, B, C, are connected with its semiaxes a, b, c by the relation $Aa^4(b^2-c^2)+Bb^4(c^2-a^2)+Cc^4(a^2-b^2)=0$, be made to roll in contact with a perfectly rough plane, the motion of this material ellipsoid will be precisely the same as that of the momental ellipsoid of the rigid body; the rough plane taking the place of the geometrical fixed plane, in contact with which the momental ellipsoid is supposed to roll. He has also investigated expressions for the pressure and friction between the ellipsoid and the rough plane, in terms of the angular velocity of the ellipsoid, and of the length of its axes, and the distance of the centre from the rough plane. In investigating independently the values of these forces, I have been led to a somewhat different treatment of the same problem, in the course of which some theorems have presented themselves which may be not without interest.

The notation which I adopt is as follows: ω_1, ω_2, ω_3 represent the component angular velocities of the ellipsoid about its principal axes, which, as proved by Professor SYLVESTER, are connected by the following equations:—

$$\frac{d\omega_1}{dt}=a^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\omega_2\omega_3,\quad \frac{d\omega_2}{dt}=b^2\left(\frac{1}{c^2}-\frac{1}{c^2}\right)\omega_3\omega_1,\quad \frac{d\omega_3}{dt}=c^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\omega_1\omega_2. \quad . \quad . \quad . \quad (1)$$

The distance from the centre of the ellipsoid to the rough plane is denoted by p, and the component angular velocity of the ellipsoid about the normal to the rough plane, which is known to be constant, by λ. The component angular velocity about the projection of the instantaneous axis on the rough plane is denoted by μ, and the whole angular velocity by ω, so that we have

$$\omega_1^2+\omega_2^2+\omega_3^2=\lambda^2+\mu^2=\omega^2. \qquad (2)$$

We have also the following relations:—

$$\frac{\omega_1^2}{a^2}+\frac{\omega_2^2}{b^2}+\frac{\omega_3^2}{c^2}=\frac{\lambda^2}{p^2}, \qquad (3)$$

$$\frac{\omega_1^2}{a^4}+\frac{\omega_2^2}{b^4}+\frac{\omega_3^2}{c^4}=\frac{\lambda^2}{p^4}. \qquad (4*)$$

And the principal moments of inertia are represented by

$$\left(\frac{G}{a^2}-\frac{H}{p^2}\right)b^2c^2,\ \left(\frac{G}{b^2}-\frac{H}{p^2}\right)c^2a^2,\ \left(\frac{G}{c^2}-\frac{H}{p^2}\right)a^2b^2,$$

which will be found, when substituted for A, B, C, to satisfy the relation already stated. In the case of a uniform ellipsoid, we have

$$G\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\frac{H}{p^2}.$$

* [The equations (1), (2), (3), (4), and the invariability of λ, may also be proved as follows. If x, y, z be the coordinates of the point of contact, referred to the principal axes, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{1}{p^2}.$$

Also

$$\frac{x}{\omega_1}=\frac{y}{\omega_2}=\frac{z}{\omega_3}=\frac{p}{\lambda},$$

whence

$$\frac{\omega_1^2}{a^2}+\frac{\omega_2^2}{b^2}+\frac{\omega_3^2}{c^2}=\frac{\lambda^2}{p^2},\quad \frac{\omega_1^2}{a^4}+\frac{\omega_2^2}{b^4}+\frac{\omega_3^2}{c^4}=\frac{\lambda^2}{p^4}.$$

Multiply these equations by $-\frac{H}{p^2}$, G, and add, then

$$a^2b^2c^2\left\{\left(\frac{G}{a^2}-\frac{H}{p^2}\right)b^2c^2\omega_1^2+\left(\frac{G}{b^2}-\frac{H}{p^2}\right)c^2a^2\omega_2^2+\left(\frac{G}{c^2}-\frac{H}{p^2}\right)a^2b^2\omega_3^2\right\}=(G-H)\frac{\lambda^2}{p^4},$$

or

$$\lambda^2=\frac{p^4}{a^2b^2c^2}\cdot\frac{\textit{vis viva}\text{ of the ellipsoid}}{G-H},$$

and is therefore constant.

Hence the direction-cosines of the perpendicular to the fixed plane are

$$\frac{\omega_1}{a^2}\frac{p^2}{\lambda},\ \frac{\omega_2}{b^2}\frac{p^2}{\lambda},\ \frac{\omega_3}{c^2}\frac{p^2}{\lambda};$$

and since this line is fixed,

$$\frac{1}{a^2}\frac{p^2}{\lambda}\frac{d\omega_1}{dt}-\frac{\omega_2}{b^2}\frac{p^2}{\lambda}\omega_3+\frac{\omega_3}{c^2}\frac{p^2}{\lambda}\omega_2=0,$$

or

$$\frac{1}{a^2}\frac{d\omega_1}{dt}-\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\omega_2\omega_3=0.$$

Similarly

$$\frac{1}{b^2}\frac{d\omega_2}{dt}-\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\omega_3\omega_1=0,$$

$$\frac{1}{c^2}\frac{d\omega_3}{dt}-\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\omega_1\omega_2=0.$$—February 1870.]

[We shall first investigate the value of h_λ, the component angular momentum of the ellipsoid about the normal to the rough plane. The cosines of the inclinations of this line to the principal axes of the ellipsoid are $\frac{p^2}{\lambda}\frac{\omega_1}{a^2}, \frac{p^2}{\lambda}\frac{\omega_2}{b^2}, \frac{p^2}{\lambda}\frac{\omega_3}{c^2}$ respectively; and the component angular momenta about the principal axes are

$$\left(\frac{G}{a^2}-\frac{H}{p^2}\right)b^2c^2\omega_1,\ \left(\frac{G}{b^2}-\frac{H}{p^2}\right)c^2a^2\omega_2,\ \left(\frac{G}{c^2}-\frac{H}{p^2}\right)a^2b^2\omega_3$$

respectively. Hence

$$h_\lambda=\frac{p^2}{\lambda}\left\{\left(\frac{G}{a^2}-\frac{H}{p^2}\right)\frac{b^2c^2}{a^2}\omega_1^2+\left(\frac{G}{b^2}-\frac{H}{p^2}\right)\frac{c^2a^2}{b^2}\omega_2^2+\left(\frac{G}{c^2}-\frac{H}{p^2}\right)\frac{a^2b^2}{c^2}\omega_3^2\right\}$$

$$=\frac{p^2}{\lambda}a^2b^2c^2\left\{G\left(\frac{\omega_1^2}{a^6}+\frac{\omega_2^2}{b^6}+\frac{\omega_3^2}{c^6}\right)-\frac{H}{p^2}\left(\frac{\omega_1^2}{a^4}+\frac{\omega_2^2}{b^4}+\frac{\omega_3^2}{c^4}\right)\right\}.$$

Now, by multiplying (2), (3), (4) by $\frac{1}{a^2b^2c^2}$, $-\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)$, $\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$ respectively, and adding, we see that

$$\frac{\omega_1^2}{a^6}+\frac{\omega_2^2}{b^6}+\frac{\omega_3^2}{c^6}=\frac{\lambda^2}{p^6}-\lambda^2\left(\frac{1}{p^2}-\frac{1}{a^2}\right)\left(\frac{1}{p^2}-\frac{1}{b^2}\right)\left(\frac{1}{p^2}-\frac{1}{c^2}\right)+\frac{\mu^2}{a^2b^2c^2}.$$

Hence

$$h_\lambda=\frac{p^2}{\lambda}\left[G\lambda^2\left\{\frac{a^2b^2c^2}{p^6}-\left(\frac{a^2}{p^2}-1\right)\left(\frac{b^2}{p^2}-1\right)\left(\frac{c^2}{p^2}-1\right)\right\}+G\mu^2-H\frac{\lambda^2}{p^6}a^2b^2c^2\right]$$

$$=(G-H)\lambda\frac{a^2b^2c^2}{p^4}+G\lambda p^2\left(1-\frac{a^2}{p^2}\right)\left(1-\frac{b^2}{p^2}\right)\left(1-\frac{c^2}{p^2}\right)+G\frac{p^2}{\lambda}\mu^2. \quad . \quad . \quad . \quad . \quad (5)$$

We shall next investigate a relation between the component angular momentum of the ellipsoid about any axis through its centre, and that, about the same axis, of a particle of mass G, situated at the point of contact of the ellipsoid and rough plane, and moving as that point moves. If l, m, n be the direction-cosines of the axis referred to the principal axes, the component angular momentum of the body about it is

$$\left(\frac{G}{a^2}-\frac{H}{p^2}\right)b^2c^2l\omega_1+\left(\frac{G}{b^2}-\frac{H}{p^2}\right)c^2a^2m\omega_2+\left(\frac{G}{c^2}-\frac{H}{p^2}\right)a^2b^2n\omega_3,$$

or

$$a^2b^2c^2\left\{G\left(\frac{l\omega_1}{a^4}+\frac{m\omega_2}{b^4}+\frac{n\omega_3}{c^4}\right)-\frac{H}{p^2}\left(\frac{l\omega_1}{a^2}+\frac{m\omega_2}{b^2}+\frac{n\omega_3}{c^2}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Again, the coordinates of the point of contact are $\frac{p}{\lambda}\omega_1, \frac{p}{\lambda}\omega_2, \frac{p}{\lambda}\omega_3$ respectively. Hence its component velocities, parallel to the axes, are

$$\frac{p}{\lambda}\frac{d\omega_1}{dt},\ \frac{p}{\lambda}\frac{d\omega_2}{dt},\ \frac{p}{\lambda}\frac{d\omega_3}{dt};$$

and its component angular momenta are therefore

$$G\frac{p^2}{\lambda^2}\left(\omega_2\frac{d\omega_3}{dt}-\omega_3\frac{d\omega_2}{dt}\right),\ G\frac{p^2}{\lambda^2}\left(\omega_3\frac{d\omega_1}{dt}-\omega_1\frac{d\omega_3}{dt}\right),\ G\frac{p^2}{\lambda^2}\left(\omega_1\frac{d\omega_2}{dt}-\omega_2\frac{d\omega_1}{dt}\right).$$

Now, by equations (1), we see that

$$\omega_2\frac{d\omega_3}{dt}-\omega_3\frac{d\omega_2}{dt}=c^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\omega_1\omega_2^2-b^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\omega_1\omega_3^2$$

$$=b^2c^2\omega_1\left\{\frac{1}{a^2}\left(\frac{\omega_2^2}{b^2}+\frac{\omega_3^2}{c^2}\right)-\left(\frac{\omega_2^2}{b^4}+\frac{\omega_3^2}{c^4}\right)\right\}$$

$$=b^2c^2\omega_1\left(\frac{1}{a^2}-\frac{1}{p^2}\right)\frac{\lambda^2}{p^2}\text{ by (3) and (4).}$$

The other component angular momenta being similarly transformed, we obtain, for the component angular momentum about the assigned axis,

$$\mathrm{G}a^2b^2c^2\left\{\frac{l\omega_1}{a^2}\left(\frac{1}{a^2}-\frac{1}{p^2}\right)+\frac{m\omega_2}{b^2}\left(\frac{1}{b^2}-\frac{1}{p^2}\right)+\frac{n\omega_3}{c^2}\left(\frac{1}{c^2}-\frac{1}{p^2}\right)\right\},$$

or

$$\mathrm{G}a^2b^2c^2\left\{\frac{l\omega_1}{a^4}+\frac{m\omega_2}{b^4}+\frac{n\omega_3}{c^4}-\frac{1}{p^2}\left(\frac{l\omega_1}{a^2}+\frac{m\omega_2}{b^2}+\frac{n\omega_3}{c^2}\right)\right\}. \quad \text{. . . . . (7)}$$

Comparing this with the expression already obtained for the angular momentum of the body, we see that the two expressions are equal if

$$\frac{l\omega_1}{a^2}+\frac{m\omega_2}{b^2}+\frac{n\omega_3}{c^2}=0,$$

i. e. if the axis be parallel to the rough plane; and generally that the angular momentum of the ellipsoid, that of the particle,

$$=(\mathrm{G}-\mathrm{H})\frac{a^2b^2c^2}{p^2}\left(\frac{l\omega_1}{a^2}+\frac{m\omega_2}{b^2}+\frac{n\omega_3}{c^2}\right),$$

which, if the axis be perpendicular to the rough plane, becomes $(\mathrm{G}-\mathrm{H})\frac{a^2b^2c^2}{p^4}\lambda$, a constant.—February 1870].

Now, let h_t denote the angular momentum of the ellipsoid about an axis through its centre, parallel to the rough plane, and at right angles to the instantaneous axis, and h_ρ about an axis through its centre parallel to the rough plane and at right angles to this last. The radius vector of the point of contact measured from the foot of the perpendicular on the rough plane is $\frac{p}{\lambda}\mu$; and hence if n denote the angular velocity of this radius vector in space, the radial and transversal component velocities of the point of contact will be $\frac{p}{\lambda}\frac{d\mu}{dt}$, $\frac{p}{\lambda}n\mu$ respectively. To obtain the component angular momenta of the particle we have only to multiply the expressions by G. Hence, by the theorem just proved,

$$h_t=\mathrm{G}\frac{p^2}{\lambda}\frac{d\mu}{dt},\ h_\rho=\mathrm{G}\frac{p^2}{\lambda}n\mu. \quad \text{. . . . . (8)}$$

We can now calculate the values of P and F, the pressure and friction of the ellipsoid against the plane. Taking moments about axes through the centre of the ellipsoid, lying in the rough plane and perpendicular to the direction of P and F respectively, each of these forces acts at an arm $\frac{p}{\lambda}\mu$. Hence

$$\mathrm{F}\frac{p}{\lambda}\mu=\frac{dh_\lambda}{dt}$$

$$=\mathrm{G}\frac{2p^2}{\lambda}\mu\frac{d\mu}{dt} \text{ by (7);}$$

$$\therefore \quad \mathrm{F}=2\mathrm{G}p\frac{d\mu}{dt}. \qquad (9)$$

It is worthy of notice that $\mathrm{F}=\frac{2\lambda}{p}h_t$.

Again,

$$\mathrm{P}\frac{p}{\lambda}\mu=\frac{dh_t}{dt}-nh_\varrho$$

$$=\mathrm{G}\frac{p^2}{\lambda}\left(\frac{d^2\mu}{dt^2}-n^2\mu\right);$$

$$\therefore \quad \mathrm{P}=\mathrm{G}p\left(\frac{1}{\mu}\frac{d^2\mu}{dt^2}-n^2\right). \qquad (10)$$

It may be desirable to replace these expressions by others in which μ shall be the only variable quantity, and which shall be free from differential coefficients. This may be done as follows. Writing, for shortness, α, β, γ in place of

$$1-\frac{a^2}{p^2},\ 1-\frac{b^2}{p^2},\ 1-\frac{c^2}{p^2}$$

respectively, it may be proved, from equations (1), (2), (3), (4), that

$$\left(\mu\frac{d\mu}{dt}\right)^2=-(\mu^2+\beta\gamma\lambda^2)(\mu^2+\gamma\alpha\lambda^2)(\mu^2+\alpha\beta\lambda^2). \qquad (11)$$

Again, it is proved by POINSOT, 'Sur la Rotation des Corps,' p. 130 (see also Quarterly Journal of Pure and Applied Mathematics, vol. vii. p. 74), that

$$p=\lambda+\alpha\beta\gamma\frac{\lambda^3}{\mu^2}, \qquad (12)$$

a result which also follows from (5), (6), (7), remembering that the angular momentum of the particle about the normal to the rough plane is $\mathrm{G}\frac{p^2}{\lambda^2}\mu^2 n$.

Now, differentiating (11),

$$\frac{d}{dt}\left(\mu\frac{d\mu}{dt}\right)=-(\mu^2+\gamma\alpha\lambda^2)(\mu^2+\alpha\beta\lambda^2)-(\mu^2+\alpha\beta\lambda^2)(\mu^2+\beta\gamma\lambda^2)$$
$$-(\mu^2+\beta\gamma\lambda^2)(\mu^2+\gamma\alpha\lambda^2);$$

$$\therefore \mu\frac{d^2\mu}{dt^2}=-(\mu^2+\gamma\alpha\lambda^2)(\mu^2+\alpha\beta\lambda^2)-(\mu^2+\alpha\beta\lambda^2)(\mu^2+\beta\gamma\lambda^2)-(\mu^2+\beta\gamma\lambda^2)(\mu^2+\gamma\alpha\lambda^2)$$
$$+\frac{(\mu^2+\beta\gamma\lambda^2)(\mu^2+\gamma\alpha\lambda^2)(\mu^2+\alpha\beta\lambda^2)}{\mu^2},$$

and

$$n^2\mu^2=\left(\lambda\mu+\alpha\beta\gamma\frac{\lambda^2}{\mu}\right)^2;$$

$$\therefore\ \mu^2\left(\frac{1}{\mu}\frac{d^2\mu}{dt^2}-n^2\right)=-2\mu^4-(1+\beta\gamma+\gamma\alpha+\alpha\beta)\lambda^2\mu^2-2\alpha\beta\gamma\lambda^4-\alpha^2\beta^2\gamma^2\frac{\lambda^6}{\mu^2},$$

$$\therefore\ \mathrm{P}=\mathrm{G}p\left\{-2\mu^2-(1+\beta\gamma+\gamma\alpha+\alpha\beta)\lambda^2-2\alpha\beta\gamma\frac{\lambda^4}{\mu^2}-\alpha^2\beta^2\gamma^2\frac{\lambda^6}{\mu^4}\right\}. \quad . \quad . \quad . \quad . \quad (13)$$

Again,

$$\mathrm{F}=\mathrm{G}p\frac{d\mu}{dt},$$

$$=\frac{\mathrm{G}p}{\mu}\left\{-(\mu^2+\beta\gamma\lambda^2)(\mu^2+\gamma\alpha\lambda^2)(\mu^2+\alpha\beta\lambda^2)\right\}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

[The theorem contained in equations (5) and (6) may perhaps receive additional illustration by a comparison of the moments, about the principal axes, of the forces acting on the ellipsoid, and of those acting on the particle coinciding with the point of contact. Since the component angular momenta of the ellipsoid about the principal axes are $\left(\frac{\mathrm{G}}{a^2}-\frac{\mathrm{H}}{p^2}\right)b^2c^2\omega_1$, $\left(\frac{\mathrm{G}}{b^2}-\frac{\mathrm{H}}{p^2}\right)c^2a^2\omega_2$, $\left(\frac{\mathrm{G}}{c^2}-\frac{\mathrm{H}}{p^2}\right)a^2b^2\omega_3$, it follows that the moment of the forces about one of the principal axes is

$$\left(\frac{\mathrm{G}}{a^2}-\frac{\mathrm{H}}{p^2}\right)b^2c^2\frac{d\omega_1}{dt}-\left\{\left(\frac{\mathrm{G}}{b^2}-\frac{\mathrm{H}}{p^2}\right)c^2a^2-\left(\frac{\mathrm{G}}{c^2}-\frac{\mathrm{H}}{p^2}\right)a^2b^2\right\}\omega_2\omega_3,$$

or

$$\mathrm{G}\left\{\frac{b^2c^2}{a^2}\frac{d\omega_1}{dt}-\left(\frac{c^2a^2}{b^2}-\frac{a^2b^2}{c^2}\right)\omega_2\omega_3\right\}-\frac{\mathrm{H}}{p^2}\left\{b^2c^2\frac{d\omega_1}{dt}-(c^2a^2-a^2b^2)\omega_2\omega_3\right\}$$

$$=\mathrm{G}\left\{\frac{b^2c^2}{a^2}\frac{d\omega_1}{dt}-\left(\frac{c^2a^2}{b^2}-\frac{a^2b^2}{c^2}\right)\omega_2\omega_3\right\}\text{ by (1),}$$

or

$$\mathrm{G}\frac{(b^2-c^2)(c^2a^2+a^2b^2-b^2c^2)}{b^2c^2}\omega_2\omega_3,$$

a result independent of H. Now, if we refer to equation (7) we shall see that the angular momenta of the particle only differ from those of the ellipsoid by having G written in place of H; consequently the moments of the forces, since they do not involve H, must be the same for the particle and the ellipsoid. It follows of course that the moments of the forces about any other axis must be the same in both cases.

In the above investigation of the value of P, I have followed Professor SYLVESTER, in assuming that the friction acts wholly in a direction perpendicular to the instantaneous axis. The other component of the friction is necessarily indeterminate, since any force in the direction of the instantaneous axis may be combined with it, without altering its effect. I have assumed this component of the friction to be zero; if it be taken to be equal to an arbitrary force F′, the value of P above investigated must be increased by $\frac{\mathrm{F}'\lambda}{\mu}$. The values of the moments of the forces are not, of course, affected by this supposition; and if F′ be so chosen that the pressure between the ellipsoid and the rough plane may be zero, the forces acting on the body will become absolutely identical with those acting on the particle G,—that is, we shall have $F'=G\frac{p}{\lambda}\left(\frac{d^2\mu}{dt^2}-n^2\omega\right)$, and F, as before, $=2Gp\frac{d\mu}{dt}$.—February 1870.]

It may be worth while to point out that the correlated and contrarelated bodies treated of in the latter part of Professor SYLVESTER's paper include, as a particular case, POINSOT's "rolling and sliding cone;" for the equation of that cone is

$$\frac{x^2}{a^2-p^2}+\frac{y^2}{b^2-p^2}+\frac{z^2}{c^2-p^2}=0,$$

which is asymptotic to the two following surfaces:—

$$\frac{x^2}{a^2-p^2}+\frac{y^2}{b^2-p^2}+\frac{z^2}{c^2-p^2}=1,$$

$$\frac{x^2}{p^2-a^2}+\frac{y^2}{p^2-b^2}+\frac{z^2}{p^2-c^2}=1,$$

the former of which is confocal, the latter contrafocal, to the momental elli soid of the free body. Hence, since the difference between the squares on corresponding semiaxes is in this case p^2, each of these hyperboloids will roll on the invariable plane through the fixed point, which will be asymptotic to it, while the plane itself rotates with uniform angular velocity λ. Hence the asymptotic cone will move in exactly the same manner.